

The Möbius Inversion Formula

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Möbius μ -function

Def: for a positive integer n , define $\mu(n)$ as follows -

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } p^r | n \text{ for some prime } p \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k, \text{ where } p_i \text{'s are distinct primes.} \end{cases}$$

e.g. ① $n=6 = 2 \times 3 = p_1 \cdot p_2$.

$$\therefore \mu(6) = (-1)^2 = 1$$

② $n=25, 5^2 | 25, \therefore \mu(25) = 0$

③ $\mu(100) = 0 \text{ or } 2^2 | 100 \text{ etc.}$

Theorem: The Möbius μ -function is multiplicative.

Proof: Let m, n be two integers s.t. $\gcd(m, n) = 1$

Then two cases $\phi(mn) = \phi(m)\phi(n)$

If $p_1^r | m$ or $p_2^s | n$ for some prime p_1 and p_2

Then $p_1^r | mn$ or $p_2^s | mn$

Then $\phi(mn) = 0$ and $\phi(m)\phi(n) = 0$ so

$$\phi(mn) = \phi(m)\phi(n)$$

If $m = p_1 p_2 \cdots p_r$ $n = q_1 q_2 \cdots q_s$ where p_i 's and q_j 's are distinct primes

Then $m n = p_1 p_2 \dots p_k q_1 q_2 \dots q_\ell$ - total $(k+\ell)$ primes.

$$\text{Then } \phi(mn) = (-1)^{k+\ell}$$

$$\text{and } \phi(m) = (-1)^k, \quad \phi(n) = (-1)^\ell$$

$$\therefore \phi(mn) = (-1)^k (-1)^\ell = \underline{\phi(m)\phi(n)}$$

Now let us see what is the sum of $\mu(d)$ when d is a divisor of n .

$$\textcircled{i} \text{ if } n=1, \text{ then } \sum_{d|1} \mu(d) = \mu(1) = 1$$

$$\textcircled{ii} \text{ if } n > 1 \text{ then let } F(n) = \sum_{d|n} \mu(d)$$

$$\text{let } n = p^k \text{ then}$$

$$\begin{aligned} F(n^k) &= \sum_{d|n^k} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \dots + \mu(p^k) \\ &= \mu(1) + \mu(p) + 0 + 0 + \dots + 0 \\ &= 1 + (-1) = 0 \end{aligned}$$

$$\text{If } n = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} \text{ then}$$

$$F(n) = F(p_1^{k_1})F(p_2^{k_2}) \dots F(p_n^{k_n}) = 0$$

[note that μ is multiplicative
 $\Rightarrow F$ is multiplicative]

So we have -

Theorem:- For each the integer $n > 1$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Möbius Inversion Formula:-

Theorem:- Let F and f be two number-theoretic functions related by the formula-

$$F(n) = \sum_{d|n} f(d)$$

Then $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$

Proof:-

$$\begin{aligned} \text{Now } \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \left\{ \sum_{c|\frac{n}{d}} f(c) \right\} \\ &= \sum_{d|n} \left\{ \sum_{c|\frac{n}{d}} \mu(d) f(c) \right\} \end{aligned} \longrightarrow ①$$

Now $d|n$ and $c|\frac{n}{d}$ implies -

$$n = kd \quad \text{and} \quad \frac{n}{d} = k'c \Rightarrow k = \frac{n}{d} = k'c$$

$$\Rightarrow n = dk'k \quad \text{and} \quad \frac{n}{c} = k'd$$

i.e. $c|n$ and $d|\frac{n}{c}$

$$\text{i.e. } d|n, c|\frac{n}{d} \Leftrightarrow c|n, d|\frac{n}{c}$$

$$\begin{aligned} \text{Therefore } \sum_{d|n} \left\{ \sum_{c|\frac{n}{d}} \mu(d) f(c) \right\} &= \sum_{c|n} \left\{ \sum_{d|\frac{n}{c}} f(c) \mu(d) \right\} \\ &= \sum_{c|n} \left\{ f(c) \sum_{d|\frac{n}{c}} \mu(d) \right\} \end{aligned}$$

Now $\sum_{d|\frac{n}{c}} \mu(d)$ vanish except $\frac{n}{c}=1$ i.e. $n=c$

$$\therefore ② \Rightarrow \sum_{c|n} \left(f(c) \sum_{d|\frac{n}{c}} \mu(d) \right) = \sum_{c=n} f(c) \cdot 1 = f(n)$$

$$\therefore \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = f(n).$$

As we know that

$$\tau(n) = \sum_{d|n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d|n} d$$

Therefore using Möbius Inversion formulae we get

$$1 = \sum_{d|n} \mu\left(\frac{n}{d}\right) \tau(d) \quad \text{and} \quad n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma(d)$$

Theorem: If F is a multiplicative and

$$F(n) = \sum_{d|n} f(d), \text{ then } f \text{ is also multiplicative.}$$

Pf:- Let m, n be two integers s.t $\gcd(m, n) = 1$. Then if $d | mn$ then $d = d_1 d_2$ s.t $d_1 | m$, $d_2 | n$ and $\gcd(d_1, d_2) = 1$.

Using Inversion formulae we have

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

$$\begin{aligned} f(mn) &= \sum_{d|mn} \mu(d) F\left(\frac{mn}{d}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1, d_2) F\left(\frac{m}{d_1} \frac{n}{d_2}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1) \mu(d_2) F\left(\frac{m}{d_1}\right) F\left(\frac{n}{d_2}\right) \\ &= \sum_{d_1|m} \mu(d_1) F\left(\frac{m}{d_1}\right) \sum_{d_2|n} \mu(d_2) F\left(\frac{n}{d_2}\right) \\ &= f(m) f(n) \end{aligned}$$

somewhere. Subsequently, it has been shown that there is a counterexample to the Mertens conjecture for at least one $n \leq (3.21)10^{64}$.

PROBLEMS 6.2

1. (a) For each positive integer n , show that

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$$

- (b) For any integer $n \geq 3$, show that $\sum_{k=1}^n \mu(k!) = 1$.

2. The Mangoldt function Λ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \text{ where } p \text{ is a prime and } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Prove that $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = - \sum_{d|n} \mu(d) \log d$.

[Hint: First show that $\sum_{d|n} \Lambda(d) = \log n$ and then apply the Möbius inversion formula.]

3. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d|n} \mu(d)f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r))$$

[Hint: By Theorem 6.4, the function F defined by $F(n) = \sum_{d|n} \mu(d)f(d)$ is multiplicative; hence, $F(n)$ is the product of the values $F(p_i^{k_i})$.]

4. If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, use Problem 3 to establish the following:

- $\sum_{d|n} \mu(d)\tau(d) = (-1)^r$.
- $\sum_{d|n} \mu(d)\sigma(d) = (-1)^r p_1 p_2 \cdots p_r$.
- $\sum_{d|n} \mu(d)/d = (1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_r)$.
- $\sum_{d|n} d\mu(d) = (1 - p_1)(1 - p_2) \cdots (1 - p_r)$.

5. Let $S(n)$ denote the number of square-free divisors of n . Establish that

$$S(n) = \sum_{d|n} |\mu(d)| = 2^{\omega(n)}$$

where $\omega(n)$ is the number of distinct prime divisors of n .

[Hint: S is a multiplicative function.]

6. Find formulas for $\sum_{d|n} \mu^2(d)/\tau(d)$ and $\sum_{d|n} \mu^2(d)/\sigma(d)$ in terms of the prime factorization of n .

7. The Liouville λ -function is defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^{k_1+k_2+\cdots+k_r}$, if the prime factorization of $n > 1$ is $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. For instance,

$$\lambda(360) = \lambda(2^3 \cdot 3^2 \cdot 5) = (-1)^{3+2+1} = (-1)^6 = 1$$

- (a) Prove that λ is a multiplicative function.